COMPARING TOPOLOGIES ON THE MORSE BOUNDARY AND QUASI-ISOMETRY INVARIANCE

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ABSTRACT. We compare several topologies on the Morse boundary $\partial_M Y$ of a CAT(0) cube complex Y. In particular, we show that the two topologies introduced by Cashen and Mackay are not equal in general and provide a new description of one of them in the language of cube complexes. As a corollary, we obtain a new approach to tackle the question whether the visual topology induces a quasi-isometry-invariant topology on the Morse boundary. This leads to an obstruction to quasi-isometry-invariance in terms of the behaviour of geodesics under quasi-isometries.

CONTENTS

1. INTRODUCTION

Boundaries at infinity are a common tool in the study of large scale geometric properties. When a group acts geometrically on a metric space, one can study the relations between the group and the boundary of the metric space. A particularly fruitful instance of this is the case of a Gromov-hyperbolic metric space and its visual boundary. Since any quasi-isometry between hyperbolic metric spaces induces a homeomorphism on the visual boundary, we can define the visual boundary of a hyperbolic group as a topological space up to homeomorphism (see [Gro87]). This is no longer true, when the group is acting on a non-positively curved space, e. g. a CAT(0) space. In [CK00], Croke and Kleiner provided an example of a group acting geometrically on two different quasi-isometric CAT(0) spaces which have non-homeomorphic visual boundaries. So we cannot associate the visual boundary as a topological space up to homeomorphism to a group.

Charney and Sultan introduced an alternative boundary for CAT(0) spaces, the contracting boundary, whose homeomorphism class is invariant under quasi-isometry (see [CS15]). Cordes generalised their concept to proper geodesic metric spaces Y, introducing the Morse boundary, denoted $\partial_M Y$ (see [Cor17]). In either case, this boundary consists of all points in the visual boundary, which are represented by geodesic rays that have similar properties to geodesic rays in hyperbolic spaces, indicating that these are the 'hyperbolic' directions in the space under consideration (see section 2 for definitions).

Charney and Sultan equipped the contracting boundary with a quasi-isometry invariant topology by using a direct limit construction. A similar construction is done to define a quasi-isometry-invariant topology on the Morse boundary in greater generality (see [CH17]). This allows us to define the Morse boundary of a group as the homeomorphism class of such a direct limit. However, this topology is not first countable in general. Concretely, this topology is not first countable for the group $\mathbb{Z}^2 * \mathbb{Z}$, an example also known as the 'tree of flats' (see [Mur18]).

Cashen and Mackay introduced two new, coarser topologies ([CM18]). Both topologies are based on the notion of fellow-traveling paths, a useful concept in the study of the visual boundary of hyperbolic spaces. Cashen and Mackay generalized this concept to proper, geodesic metric spaces to introduce the topology of fellow-traveling geodesics $\mathcal{FG}$ and the topology of fellow-traveling quasi-geodesics $\mathcal{FQ}$, the second of which is invariant under quasi-isometries and thus allows us to turn the Morse boundary into a topological invariant of a group (see section 2.1 for definitions of the Morse boundary, $\mathcal{FG}$ and $\mathcal{FQ}$). If a group acts geometrically on a proper geodesic metric space Y, then $(\partial_M Y, \mathcal{FQ})$ is metrizable and second countable. However, Cashen and Mackay's proof of metrizability relies on the Urysohn metrisation theorem, whose proofs do not provide a metric that is easy to compute. Giving an explicit construction of a metrization of $\mathcal{FQ}$ that stays faithful to the geometric context of Morse boundaries is still an open problem at the time of writing.

In this article, we restrict our attention to the Morse boundary of CAT(0) cube complexes. Cube complexes were introduced by Gromov in [Gro87] and have become a central object in geometric group theory over the last decade due to their fruitfulness in solving problems in group theory and low-dimensional topology and due to the fact that many interesting groups are cubulable, i. e. they act properly and cocompactly on a CAT(0) cube complex. The class of groups that are cubulable includes Right-angled Artin groups, hyperbolic 3-manifold groups ([BW12]), most non-geometric 3-manifold groups ([PW14], [HP15], [PW18]), small cancelation groups ([Wis04]) and many others. Cubulated groups played a key role in Agol's and Wise's proof of the virtual Haken and virtual fibered conjecture ([Wis11], [Ago13]).

When considering a CAT(0) cube complex Y, one can define the following topology on the visual boundary: Fix a vertex $o \in Y$ as a base point and let $h_1, \dots, h_n$ be distinct hyperplanes in Y. Denote the visual boundary of Y by $\partial_\infty Y$. Define the set

$$V_{o,h_1,\dots,h_n} := \{\xi \in \partial_\infty Y | \text{The unique geodesic representative of } \xi \text{ based at } o \text{ crosses the hyperplanes } h_1, \dots, h_n.\}.$$

The collection $\{V_{o,h_1,\dots,h_n}\}_{n,h_1,\dots,h_n}$ forms the basis of a topology on $\partial_\infty Y$. We denote the subspace topology on the Morse boundary induced by this topology by $\mathcal{HYP}$.

A comparison of $\mathcal{FG}$ and $\mathcal{HYP}$ yields

Theorem 1.1. *Let Y be a uniformly locally finite* CAT(0) *cube complex. Then the topologies $\mathcal{FG}$ and $\mathcal{HYP}$ coincide on the Morse boundary.*

It is shown in [CM18] that $\mathcal{FG} \subset \mathcal{FQ}$ and a sufficient condition for equality is given. We show that equality does not hold in general.

Theorem 1.2. *There exists a uniformly locally finite* CAT(0) *cube complex Y that admits a geometric action by a group and $\mathcal{FG} \neq \mathcal{FQ}$ on $\partial_M Y$.*

The counter-example used to prove Theorem 1.2 is the same Right-angled Artin group that was already used by Croke and Kleiner to show that the visual boundary is not a quasi-isometry invariant for CAT(0) spaces (see section 3 for details). Our construction to prove that this is a counter-example shows in particular that no minor adjustments to Cashen and Mackays concept of fellow-traveling will change the inequality of $\mathcal{FG}$ and $\mathcal{FQ}$.

Combining Theorem 1.1 with results from [CM18] and [BF18a] (see section 5 for details), we obtain

Corollary 1.3. *Let Y be a uniformly locally finite* CAT(0) *cube complex. Then the following topologies on $\partial_M Y$ coincide:*

(1) The subspace topology induced by the visual topology
(2) $\mathcal{FG}$
(3) $\mathcal{HYP}$
(4) The topology induced by the Roller boundary.

Note that, on $\partial_\infty Y$, the visual topology and the topology generated by the sets $V_{o,h_1,\dots,h_n}$ do not coincide in general; $\mathbb{R}^2$ with its standard cubulation provides an easy counter example.

In [Cas16], Cashen showed that the subspace topology induced by the visual topology is not invariant under quasi-isometries in general. Since his counter examples do not admit a cocompact action by isometries, he raised the question whether the Morse boundary with the visual topology is invariant under quasi-isometry when the spaces in question admit a cocompact action by isometries. Corollary 1.3 provides us with a new tool to study this question for cubulable groups.

One necessary condition for the quasi-isometry invariance of $\mathcal{FG}$ is that it is independent of the choice of base point. It turns out that $\mathcal{FG}$ (and thus the other three topologies in Corollary 1.3) are independent of the choice of base point if the metric space Y is CAT(0). We provide a counter example for non-CAT(0) spaces. In fact, our counter example will be a finitely generated small cancellation group.

Restricting our attention to a special class of CAT(0) cube complexes (which includes the universal covering of the Salvetti complex of any RAAG) allows us to give an explicit construction of a metric on $\partial_M Y$ that induces the topology $\mathcal{FG}$. This metric depends on the choice of a base point. We show that the cross ratio induced by this metric also depends on the choice of a base point and we will compare these cross ratios to the one introduced in [BFIM18].

The remainder of the paper is organised as follows. In section 2, we recall basic notions and facts about Morse boundaries, CAT(0) cube complexes and RAAGs. We prove that for CAT(0) spaces, $\mathcal{FG}$ is independent of the choice of base point and provide a counter example for non-CAT(0) spaces. In section 3, we will prove Theorem 1.1 and Theorem 1.2. In section 4, we introduce a metric on $\partial_M Y$ that induces the topology $\mathcal{FG}$ for a special class of CAT(0) cube complexes and compare this metric to notions introduced in [BFIM18]. In section 5, we will discuss Corollary 1.3 and finish with a short analysis of the question whether $\mathcal{HYP}$ is invariant under quasi-isometries.

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2. Preliminaries

2.1. **The Morse boundary.** For a more thorough introduction to Morse boundaries and their properties, see [Cor17].

Let (Y, d) be a proper, geodesic metric space. Given a subset $S \subset Y$ and $R \geq 0$, we denote the *R-neighbourhood* of S by

$$N_R(S) := \{y \in Y | \exists s \in S : d(y, s) \leq R\}.$$

The Hausdorff distance between two subsets $S, S' \subset Y$ is defined to be $d_{Haus}(S, S') := \inf\{r | S \subset N_r(S'), S' \subset N_r(S)\}$. Two quasi-geodesic rays γ, γ' are called asymptotically equivalent, if they have finite Hausdorff distance.

Suppose Y is a CAT(0) space. The visual boundary $\partial_\infty Y$ of (Y, d) is defined to be the set of all equivalence classes of geodesic rays. Equivalently, one may fix a base point o and consider equivalence classes of geodesic rays that start at o.

We now drop the assumption that Y is CAT(0). Let $N : \mathbb{R}_{\geq 1} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a function. A set $S \subset Y$ is called *N-Morse* if $\forall K \geq 1$, $\forall C \geq 0$, every (K, C)-quasi-geodesic γ whose endpoints lie in S is contained in the $N(K, C)$-neighbourhood of S, i. e. $\gamma \subset N_{N(K,C)}(S)$. We call a set Morse, if there exists a function N, such that the set is N-Morse. We define the *Morse boundary* $\partial_M Y$ to be the set of all equivalence classes of Morse geodesic rays. Note that we could instead consider the set of all equivalence classes of Morse quasi-geodesic rays. By Lemma 5.2 in [CM18], every Morse quasi-geodesic ray is asymptotically equivalent to a Morse geodesic ray. This implies that every Morse quasi-geodesics represents a point in the Morse boundary and we will think of Morse quasi-geodesics as representing points in $\partial_M Y$.

There is an equivalent characterisation of the Morse-property. Let $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a non-decreasing, eventually non-negative function, which is sublinear, i. e. $\lim_{r\to\infty} \frac{\rho(r)}{r} = 0$. Consider a closed set $S \subset Y$ and $y \in Y$. We denote the set of closest points to y in S by $\pi_S(y)$ and call π_S the *closest point projection onto* S, even though π_S is, strictly speaking, not a map from Y to S. A closed set $S \subset Y$ is called *ρ-contracting*, if for all $x, y \in Y \setminus S$ with $d(x, y) < d(S, y)$, we have that $\mathrm{diam}(\pi_S(x), \pi_S(y)) \leq \rho(d(S, y))$. In other words, for any point $y \in Y \setminus S$, the projection of the largest open ball B, centered at y that does not intersect S, onto S has diameter bounded by $\rho(r)$ where r denotes the radius of B. We call a closed set sublinearly contracting, if it is ρ-contracting for some non-decreasing, eventually non-negative, sublinear function ρ.

In [ACGH17], the authors proved that for every function N, there exists a function ρ, depending only on N, such that any closed N-Morse set is ρ-contracting. Conversely, for every ρ there exists an N, such that every closed ρ-contracting set is N-Morse (cf. Theorem 1.4, Proposition 4.1 and Proposition 4.2 in [ACGH17]). Thus, we see that the contracting boundary, which is defined to be the set of all equivalence classes of quasi-geodesics, that admit a ρ-contracting representative, is the same as the Morse boundary. If Y is CAT(0), a geodesic is Morse if and only if there exists a constant D such that the geodesic is D-contracting (cf. [BF09, Sul14]). Therefore, we see that in CAT(0) spaces, geodesics are sublinearly contracting if and only if they are contracting.

Using the equivalence between Morse and sublinearly contracting, we obtain the following results. By Lemma 6.3 in [ACGH17], any set in Y that has bounded Hausdorff-distance from an N-Morse set is N'-Morse for some function N' which depends only on N and the Hausdorff-distance between the two sets. In particular, all representatives of a point in the Morse boundary are Morse. By Proposition 2.4 in [Cor17], we know that for any N-Morse geodesic ray γ and any geodesic ray γ' asymptotic to γ, their Hausdorff-distance is bounded from above by a constant depending only on N and the distance of their starting points. In particular, for any $\xi \in \partial_M Y$ and any bounded set $B \subset Y$, there exists some N, such that all geodesic representatives of ξ that start in B are N-Morse.

Note that so far, we have only introduced the Morse boundary as a set. We will now recall some constructions and results from [CM18] in order to define the topologies $\mathcal{FG}$ and $\mathcal{FQ}$. We begin with their key-observation on the divergence-behaviour of quasi-geodesics from sublinearly contracting sets. Given a sublinear function ρ, $K \geq 1$ and $C \geq 0$, we define

$$\kappa(\rho, K, C) := \max(3K^2, 3C, 1 + \inf\{R \,|\, \forall r \geq R, \rho(r) \leq 3K^2 r\})$$

and

$$\kappa'(\rho, K, C) := (K^2 + 2)(2\kappa(\rho, K, C) + C).$$

Proposition 2.1 (Corollary 4.3 in [CM18])**.** *Let Z be ρ-contracting and let β be a continuous (K, C)-quasi-geodesic ray with $d(\beta_0, Z) \leq \kappa(\rho, K, C)$. There are two possibilities:*

1. *The set $\{t | d(\beta_t, Z) \leq \kappa(\rho, K, C)\}$ is unbounded and β is contained in the $\kappa'(\rho, K, C)$-neighbourhood of Z.*
2. *There exists a last time T_0 such that $d(\beta_{T_0}, Z) = \kappa(\rho, K, C)$ and:*

$$\forall t, d(\beta_t, Z) \geq \frac{1}{2K}(t - T_0) - 2(C + \kappa(\rho, K, C)).$$

This motivates the following definition. Fix $o \in Y$. Let $\xi \in \partial_M Y$. Since ξ is in the contracting boundary, there exists ρ such that all geodesic representatives of ξ based at o are ρ-contracting. Choose one geodesic representative $\gamma \in \xi$ that is based at o. Let $R \geq 0$. We say that a (K, C)-quasi-geodesic β fellow-travels along γ for distance R, if $\beta \cap N_{\kappa(\rho,K,C)}(\gamma \setminus B_R(o)) \neq \emptyset$, where $B_R(o)$ denotes the open ball of radius R centered at o. We define the set

$$U_{o,R}(\xi) := \{\eta \in \partial_M Y | \text{ For all } \rho, K, C, \text{ such that } \gamma \text{ is } \rho\text{-contracting, all } (K, C)\text{-quasi-geodesic representatives of } \eta \text{ that are based at } o \text{ fellow-travel along } \gamma \text{ for distance } R.\}.$$

Thinking of geodesics as $(1, 0)$-quasi-geodesics, we analogously define

$$V_{o,R}(\xi) := \{\eta \in \partial_M Y | \text{ For all } \rho, \text{ such that } \gamma \text{ is } \rho\text{-constracting, all geodesic representatives of } \eta \text{ that are based at } o \text{ fellow-travel along } \gamma \text{ for distance } R.\}.$$

Cashen and Mackay show that the two families $\{U_{o,R}(\xi)\}_{R,\xi}$ and $\{V_{o,R}(\xi)\}_{R,\xi}$ are neighbourhood bases of two topologies (Proposition 5.5 and 7.1 in [CM18]). We denote the topology induced by $\{U_{o,R}(\xi)\}_{R,\xi}$ by $\mathcal{FQ}$ and the topology induced by $\{V_{o,R}(\xi)\}_{R,\xi}$ by $\mathcal{FG}$. In other words, a set U is open in $\mathcal{FQ}$ if and only if for every point $\xi \in U$ there exists some $R > 0$, such that $U_{o,R}(\xi) \subset U$. The analogous statement defines $\mathcal{FG}$.

Proposition 2.2. *$\mathcal{FQ}$ is independent of the choice of the base point o. If Y is* CAT(0)*, then $\mathcal{FG}$ is also independent of the choice of o.*

While the first part of this proposition was proven in Proposition 5.11 of [CM18], we need to provide a proof for the second part. Before we do so, we state a key technical lemma from [CM18].

Lemma 2.3 (Lemma 4.6 of [CM18])**.** *Let α be a ρ-contracting geodesic ray and let β be a continuous (K,C)-quasi-geodesic ray with $\alpha(0) = \beta(0) = o$. Given some R and J, suppose there exists a point $x \in \alpha$ with $d(x,o) \geq R$ and $d(x,\beta) \leq J$. Let y be the last point on the subsegment of α between o and x such that $d(y,\beta) = \kappa(\rho,K,C)$. There is a constant $M \leq 2$ and a function $\lambda(\phi,p,q)$ defined for sublinear ϕ, $p \geq 1$ and $q \geq 0$ such that λ is monotonically increasing in p and q and*

$$d(x,y) \leq MJ + \lambda(\rho,K,C).$$

Thus,

$$d(o,y) \geq R - MJ - \lambda(\rho,K,C).$$

We reformulate this Lemma in a way that will be a bit more handy to use.

Corollary 2.4. *Let α be a ρ-contracting geodesic ray based at o, $K \geq 1$, $C \geq 0$, $J \geq 0$, $r \geq 0$. Then, there exists a constant $R(\rho,K,C,r,J)$ such that every (K,C)-quasi-geodesic ray β based at o satisfying $(*)$ is fellow-traveling along α for at least distance r.*

- $(*)$ *There exists a point $p \in \alpha$ satisfying $d(o,p) \geq R(\rho,K,C,r,J)$ and $d(\beta,p) \leq J$.*

This corollary is obtained by thinking of $r = d(o,y)$. Then, Lemma 2.3 implies that, if we find $p \in \alpha$ such that $d(o,p) \geq R(\rho,K,C,r,J) := r + 2J + \lambda(\rho,K,C)$, then we find $y \in \alpha$ such that $d(o,y) \geq r$, implying that β fellow-travels along α for distance r.

Note that condition $(*)$ itself can be thought of as a version of fellow-traveling, where the quasi-geodesic β is J-fellow-traveling along the geodesic α for distance $R = R(\rho,K,C,r,J)$ iff $\beta \cap N_J(\alpha \setminus B_o(R)) \neq \emptyset$.

Proof of Proposition 2.2. For $\mathcal{FQ}$, this has been proven in [CM18]. Suppose that Y is CAT(0). Let o, $o' \in Y$, $\xi \in \partial_M Y$. There exists a sublinear function ρ, such that all geodesic representatives of ξ based at o or o' are ρ-contracting. To distinguish the neighbourhood-bases with respect to o and

o', we will denote the neighbourhoods by $U_{o,R}(\xi)$ and $U_{o',R}(\xi)$ respectively. Let $\alpha \in \xi$ be a geodesic based at o and $\alpha' \in \xi$ a geodesic based at o'.

Let $R \geq 0$. We need to find R' such that $U_{o',R'}(\xi) \subset U_{o,R}(\xi)$. Put

$$R' := R\left(\rho, 1, 0, R, [\kappa'(\rho, 1, d(o,o')) + \kappa(\rho, 1, 0) + d(o,o')]\right) + \kappa'(\rho, 1, d(o,o')) + d(o,o')$$

using the function $R(\cdot,\cdot,\cdot,\cdot,\cdot)$ from Corollary 2.4. We claim that $U_{o',R'}(\xi) \subset U_{o,R}(\xi)$.

Let $\eta \in U_{o',R'}(\xi)$, $\beta \in \eta$ a geodesic representative based at o and $\beta' \in \eta$ a geodesic representative based at o'. We know that $\beta' \cap N_{\kappa(\rho,1,0)}(\alpha' \setminus B_{R'}(o)) \neq \emptyset$ and thus, there exist points $p' \in \alpha' \setminus B_{R'}(o)$ and $q' \in \beta'$ such that $d(p', q') \leq \kappa(\rho, 1, 0)$. Since Y is CAT(0) and β and β' are asymptotically equivalent, the distance function $d(\beta(t), \beta'(t))$ is convex and bounded. We conclude that it is bounded by $d(o, o')$. In particular, we find $q \in \beta$ with $d(o,q) = d(o', q')$ and $d(q, q') \leq d(o, o')$. Consider a geodesic δ from o to o'. The concatenation $\delta * \alpha'$ is a $(1, d(o,o'))$-quasi-geodesic representative of ξ, based at o. By Corollary 4.3 in [CM18], every point in $\delta * \alpha'$ is $\kappa'(\rho, 1, d(o,o'))$-close to α. In particular, there exists a point $p \in \alpha$ such that $d(p, p') \leq \kappa'(\rho, 1, d(o,o'))$ and

$$\begin{aligned} d(o,p) &\geq d(o', p') - \kappa'(\rho, 1, d(o,o')) - d(o,o') \\ &\geq R(\rho, 1, 0, R, \kappa'(\rho, 1, d(o,o')) + \kappa(\rho, 1, 0) + d(o,o')). \end{aligned}$$

Overall, we find that $d(p,q) \leq \kappa'(\rho, 1, d(o,o')) + \kappa(\rho, 1, 0) + d(o,o')$ and $d(o,p) \geq R(\rho, 1, 0, R, \kappa'(\rho, 1, d(o,o')) + \kappa(\rho, 1, 0) + d(o,o'))$. Using Corollary 2.4 with $J = \kappa'(\rho, 1, d(o,o')) + \kappa(\rho, 1, 0) + d(o,o')$, we conclude that β is fellow-traveling along α for distance at least R. Since β was an arbitrary geodesic representative of $\eta \in U_{o',R'}(\xi)$ based at o, we conclude that $U_{o',R'}(\xi) \subset U_{o,R}(\xi)$. By symmetry, this implies that both neighbourhood-bases induce the same topology $\mathcal{FG}$.

□

Remark 2.5. Note that the assumption that Y is CAT(0) is only used to find an upper bound for $d_{Haus}(\beta, \beta')$ that does not depend on the contracting functions of β and β'. This is a weaker assumption than being CAT(0). For example, it is sufficient, if the distance function $d(\beta(t), \beta'(t))$ is convex for any two geodesics β, β' in Y. Spaces with this convexity property are also called *Busemann spaces*. Since we will focus on CAT(0) spaces in the rest of the paper, we will not discuss this more general situation.

Note that $\mathcal{FG}$ is not base point invariant in general. What follows is a counter example.

Example 2.6. Let R be a geodesic ray (i. e. a copy of $[0, \infty)$). For $i \geq 1$, attach geodesic rays R_i to the points $i \in R$. We will distinguish between the real number x in R and the same real number in R_i by writing $R(x)$ and $R_i(x)$ respectively. Further, we denote $o := R(0)$. Let f be a superlinear function, i. e. $\lim_{r \to \infty} \frac{r}{f(r)} = 0$, such that f is injective and $f(i) > i$ for all $i \in \mathbb{N}$. Glue intervals of length $1 + i + 6f(i)$ into our space by attaching their endpoints to o and $R_i(6f(i))$. Denote these intervals by S_i. For $0 \leq i \leq j$, denote the interval of length i in S_j that starts at o by $I_{i,j}$. For i fixed, we glue together $I_{i,j}$ for all $j \geq i$. We denote this space by Y and denote the point in all S_i of distance 1 from o by o' (all these points have been identified by the gluing). The length of each ray that we used to build Y induces a shortest-path-metric on Y.

We claim that R is a ρ-contracting geodesic ray. Indeed, a ball in $Y \setminus R$, centered at a point $p \in R_i \cup S_i$ with $d(p, R) \geq 3f(i)$ gets projected into the set $\{o, R(i)\}$ which has diameter i, while a ball centered at a point $p \in R_i \cup S_i$ with $d(p, R) \leq 3f(i)$ is sent to a single point. Therefore, R is ρ-contracting for $\rho = \frac{1}{3} \cdot f^{-1}$. Since f is superlinear, its inverse function is sublinear. Further, all R_i are sublinearly contracting as well, because the projection of $Y \setminus R_i$ onto R_i sends everything onto two points in R_i. We write $\kappa := \kappa(\rho, 1, 0)$.

Consider the neighbourhoods of $[R] \in \partial_M Y$ in $\mathcal{FG}$ with respect to o and o'. With respect to o, the (unique) geodesic representative of $[R_i]$ stays κ-close to R for distance $i + \kappa$. Thus, for every $r > 0$, $U_{o,r}([R])$ contains $[R_i]$ for all $i > r - \kappa$.

With respect to o', however, the geodesic representative of $[R_i]$ stays κ-close to the geodesic representative of $[R]$ only for distance κ. Thus, $U_{o',r}([R]) = \{[R]\}$ for $r > \kappa$. We conclude that $\mathcal{FG}$ is not base point independent.

One may ask whether $\mathcal{FG}$ is base point invariant when the space Y admits a cocompact group action by isometries. It turns out that this is not the case either. The geometry displayed above can be embedded into a finitely generated, infinitely presented small cancellation group. This is done as follows: The space Y above can be equipped with the structure of a graph all whose edges have length 1. From now on, we consider Y equipped with this graph structure. We can label the edges of this graph with letters of the alphabet $S := \{a, b_1, b_2, b_3, b_4, b_5, b_6, c, d_1, d_2, d_3, d_4, d_5, d_6\}$ such that:

(1) the geodesic ray R spells the word a^∞,
(2) the geodesic ray R_i spells $b_1^{f(i)} b_2^{f(i)} b_3^{f(i)} b_4^{f(i)} b_5^{f(i)} b_6^{f(i)} c^\infty$,
(3) the segment S_i starting at o spells $c^i b_1 d_6^{f(i)} d_5^{f(i)} d_4^{f(i)} d_3^{f(i)} d_2^{f(i)} d_1^{f(i)}$.

This labeling allows us to embed the graph Y into the Cayley-graph $\mathrm{Cay}(G, S)$ of the group

$$G := < a, b_1, b_2, b_3, b_4, b_5, b_6, c, d_1, d_2, d_3, d_4, d_5, d_6 \quad | i \in \mathbb{N}, a^i b_1^{f(i)} b_2^{f(i)} b_3^{f(i)} b_4^{f(i)} b_5^{f(i)} b_6^{f(i)} d_1^{-f(i)} d_2^{-f(i)} d_3^{-f(i)} d_4^{-f(i)} d_5^{-f(i)} d_6^{-f(i)} b_1^{-1} c^{-i} > .$$

generated by the labeled graph Y (cf. Definition 1.1 in [Gru15]). We claim that Y isometrically embeds into $\mathrm{Cay}(G, S)$ such that R and R_i are still sublinearly contracting and the topological behaviour on the boundary remains unchanged. We first note that G is a small cancellation group. This follows from the fact that, for $i < j$, the largest shared subword of the i-th

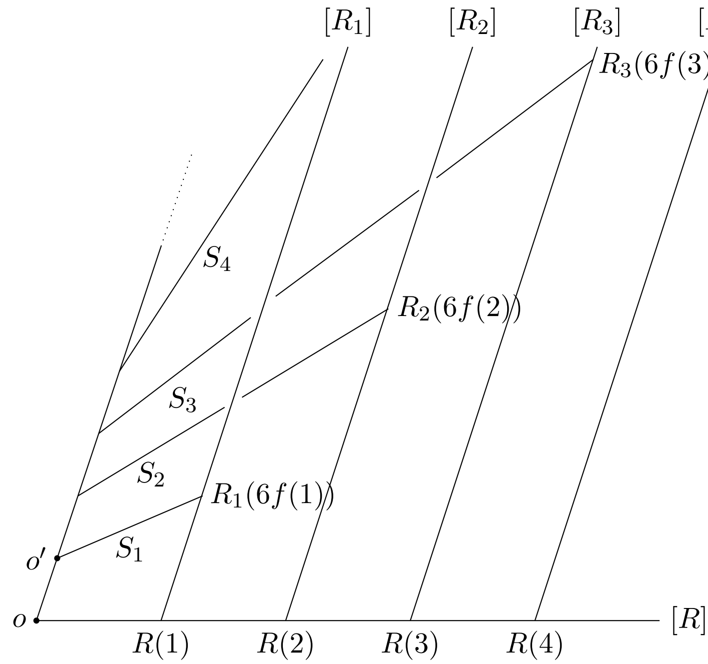

FIGURE 1. The graph Y. The beginning of the segments S_i are all identified. For every i, the segments $[o, R(i)]$, $[R(i), R_i(6f(i))]$, S_i form a cycle, where the length of $[o, R(i)]$ is significantly shorter than the length of the other two segments, since f is superlinear. This forces R to be sublinearly contracting.

and j-th relation in the presentation above has length $2f(i)$, while both relations have length greater than $12f(i)$. The same argument tells us that any two distinct, simple, closed paths γ, γ' in the labelled graph Y that share a common subpath p satisfy $l(p) \leq \frac{1}{6}l(\gamma)$. By definition, this means that Y satisfies the property $Gr'(\frac{1}{6})$ (cf. Definitions 1.2 & 1.3 in [Gru15]). By Theorem 5.10 in [Gru15], this implies that Y embeds isometrically into the Cayley-graph $\mathrm{Cay}(G, S)$ induced by Y. In particular, R and R_i are sent to geodesic rays in $\mathrm{Cay}(G, S)$. Theorem 4.1 in [ACGH18] states that for any graph that satisfies $Gr'(\frac{1}{6})$, its embedding into the Cayley-graph induced by Y has the property that a geodesic in $\mathrm{Cay}(G, S)$ is sublinearly contracting if and only if it is uniformly locally contracting in Y. Since R and R_i are sublinearly contracting in Y, we can use the theorem and conclude that their embedded images in $\mathrm{Cay}(G, S)$ are sublinearly contracting. Therefore, they induce points in the Morse boundary of $\mathrm{Cay}(G, S)$ and since the embedding of Y is isometric, the topological properties that these boundary points have in Y carry over. We conclude that $\mathcal{FG}$ exhibits the same dependence on base points in $\partial_M \mathrm{Cay}(G, S)$ as in $\partial_M Y$.

2.2. CAT(0) **cube complexes.** For an in-depth introduction to CAT(0) cube complexes, see [Sag14]. We will focus on fixing notation and recalling some definitions and facts that will be used in the remainder of the paper.

Let Y be a simply connected cube complex satisfying Gromov's no-$\triangle$-condition; see 4.2.C in [Gro87] and Chapter II.5 in [BH99]. Unless specified otherwise, all points $v \in Y$ are implicitly understood to be vertices and all subsets of Y are contained in the 0-skeleton; this applies in particular to edges and cubes. The *link* of v (denoted $\mathrm{lk}(v)$) is the simplicial complex consisting of an $(n-1)$-simplex for every n-cube of Y based at v, with the same face relations. We denote by $\deg(v)$ the number of edges of Y that contain v.

The Euclidean metrics on the cubes of Y fit together to yield a CAT(0) metric on Y. We can however also endow each cube $[0,1]^k \subseteq Y$ with the restriction of the ℓ^1 metric of $\mathbb{R}^k$ and consider the induced path metric $d_{\ell^1}(-,-)$. We refer to d as the *combinatorial metric* (or ℓ^1 *metric*). In finite dimensional cube complexes, the CAT(0) and combinatorial metrics are bi-Lipschitz equivalent and complete. In particular, if all cubes in a cube complex have dimension $\leq n$, then the CAT(0) and combinatorial metrics are $\sqrt{n}$-bi-Lipschitz equivalent.

The combinatorial metric allows us to introduce *combinatorial geodesics*, which are geodesics between vertices of Y with respect to the combinatorial metric that are fully contained in the 1-skeleton of Y. If all cubes in Y have dimension at most n, every combinatorial geodesic is a $(\sqrt{n}, 0)$-quasi-geodesic. Combinatorial geodesics are fully determined by the sequence of hyperplanes they cross.

We will refer to simply connected cube complexes satisfying Gromov's no-$\triangle$-condition as CAT(0) *cube complexes*, regardless of whether we consider it equipped with the CAT(0) metric or the combinatorial metric. In order to distinguish geodesics in the CAT(0) metric from the ℓ^1-metric, we will call the former geodesics, while the latter will only appear in the form of the combinatorial geodesics introduced above.

We call a CAT(0) cube complex Y *locally finite*, if for every vertex $v \in Y$, there are at most finitely many edges incident to v. We say Y is *uniformly locally finite* if there exists a constant ν, such that for every vertex $v \in Y$, there are at most ν many edges incident to v. We say that ν is an upper bound for the valence of all vertices in Y. We say that a CAT(0) cube complex has *uniformly bounded dimension*, if there exists a constant B such that every cube in Y has dimension at most B.

Let $\mathcal{W}(Y)$ and $\mathcal{H}(Y)$ be the set of all hyperplanes and of all halfspaces of Y respectively. Given a halfspace s, denote the other halfspace bounded by the same hyperplane by s^*. Given a hyperplane h, we call a choice of halfspace bounded by h an *orientation of* h. We sometimes denote the two orientations of h by $\{h^+, h^-\}$. The set $\mathcal{H}$ is endowed with the order relation given by inclusions; the involution $*$ is order reversing. The triple $(\mathcal{H}, \subseteq, *)$ is thus a *pocset* (see [Sag14]).

Two hyperplanes are called *transverse* if they intersect. Note that every intersection $h_1 \cap \cdots \cap h_k$ of pairwise transverse hyperplanes $h_1, \ldots, h_k$ inherits a CAT(0) cube complex structure. Its cells are precisely the intersections $h_1 \cap \cdots \cap h_k \cap c$ for any cube $c \subseteq Y$. Alternatively, $h_1 \cap \cdots \cap h_k$ can be viewed as a subcomplex of the cubical subdivision of Y.

Analogously, two halfspaces s, $s' \in \mathcal{H}$ are called *transverse*, if their bounding halfspaces are transverse. Equivalently, they are transverse if and only if the four intersections $s \cap s'$, $s \cap s'^*$, $s^* \cap s'$, $s^* \cap s'^*$ are non-empty. A hyperplane is *transverse* to a halfspace s if it is transverse to the hyperplane that bounds s.

Given two subsets U, $V \subset Y$, we define $\mathcal{W}(U, V)$ to be the set of all hyperplanes that separate U from V. Given a (combinatorial) geodesic γ, we write $\mathcal{W}(\gamma)$ for the set of hyperplanes crossed by γ.

Given a path γ in Y, the length of γ with respect to either metric can be estimated from below by $\#\mathcal{W}(\gamma) - 1$.

Let e be an edge in Y. We denote the hyperplane crossed by e by $h(e)$. We say that a hyperplane h is *adjacent* to a vertex $v \in Y$ if $h = h(e)$ for an edge incident to v.

We say that Y has *no free vertices* if for all vertices v and all edges e incident to v, there exists an edge e' incident to v, such that $h(e) \cap h(e') = \emptyset$.

Suppose, Y has no free vertices. Given an oriented edge e we can extend it to a geodesic segment in Y as follows: The orientation provides us with an endpoint v of e. Choose an edge e' incident to v such that $h(e') \cap h(e) = \emptyset$. The concatenation $e * e'$, where we interpret e and e' as paths with arc-length parametrization, is a geodesic in Y. We say that we *extend e by e'*. Given a concatenation of edges $e_1 * \cdots * e_n$ such that $h(e_i) \cap h(e_j) = \emptyset$ for all $i \neq j$, we can extend it to to a geodesic of length $n+1$ by adding an edge e_{n+1} incident to the endpoint of $e_1 * \cdots * e_n$ that does not intersect $h(e_i)$ for any $1 \leq i \leq n$. This way, we can extend any oriented edge to a geodesic of length $n+1$ by choosing a sequence of edges $e_1, \ldots, e_n$, which are mutually disjoint and do not intersect with e. We say that we extend the oriented edge (or path) e by n many steps to the geodesic $e * e_1 * \cdots * e_n$. Note that – in general – we may have several choices to extend e by n many steps. If e is contained in a flat F, i. e. an embedding of $\mathbb{R}^2$ with standard cubulation that respects the cube complex structure, we have a unique way to extend the oriented edge e by n many steps inside the flat F (since there is always a unique edge incident to the endpoint of the path that does not intersect any of the preceding hyperplanes). We say that we *extend e by n many steps in the flat F*.

Given $y \in Y$, we denote by $\sigma_y \subseteq \mathcal{H}$ the set of all halfspaces containing the point y. It satisfies the following properties:

(1) given any two halfspaces s, $s' \in \sigma_y$, we have $s \cap s' \neq \emptyset$;
(2) for any hyperplane $h \in \mathcal{W}$, a side of h lies in σ_y;
(3) every descending chain of halfspaces in σ_y is finite.

Subsets $\sigma \subseteq \mathcal{H}$ satisfying (1)–(3) are known as *DCC ultrafilters*. We refer to a set $\sigma \subseteq \mathcal{H}$ satisfying only (1) and (2) simply as an *ultrafilter*. We will also think of an ultrafilter as a map that associates to every hyperplane one of its orientations.

Let $\iota : X \to \prod_{h \in \mathcal{W}} \{h^+, h^-\}$ denote the map that takes each point y to the set σ_y. Its image $\iota(Y)$ is precisely the collection of all DCC ultrafilters. Endowing $\prod_{h \in \mathcal{W}} \{h^+, h^-\}$ with the product topology, we can consider the closure $\overline{\iota(Y)}$, which coincides with the set of all ultrafilters. Equipped with the subspace topology, this is a compact Hausdorff space known as the *Roller compactification* of Y; we denote it by $\overline{Y}$.

The *Roller boundary* $\partial_R Y$ is defined as the difference $\overline{Y} \setminus \iota(Y)$. If Y is locally finite, $\iota(Y)$ is open in $\overline{Y}$ and $\partial_R Y$ is compact; however, this is not true in general. We prefer to imagine $\partial_R Y$ as a set of points at infinity, represented by combinatorial geodesics in Y, rather than a set of ultrafilters. We will therefore write $y \in \partial_R Y$ for points in the Roller boundary and employ the notation $\sigma_y \subseteq \mathcal{H}$ to refer to the ultrafilter representing y. We say that a hyperplane h *separates* two points x and y in the Roller boundary if σ_x and σ_y do not contain the same halfspace bounded by h. In other words, they induce opposite orientation on h.

Given two points $x, y \in \partial_R Y$, we say they lie in the same *component* if and only if there are only finitely many hyperplanes that separate x from y. This defines an equivalence relation on $\partial_R Y$ and partitions the Roller boundary into equivalence classes, called components. Each component inherits the structure of a CAT(0) cube complex whose hyperplanes are a strict subset of the set of hyperplanes of Y. We say that a hyperplane $k \in \mathcal{W}(Y)$ *intersects* a component C whenever it corresponds to a hyperplane in C. Note that for any two hyperplanes h, k that intersect a component C, there exist infinitely many $h_i \in \mathcal{W}(Y)$ that intersect both h and k.

Let Y have uniformly bounded dimension. A point $x \in \partial_R Y$ is called *Morse* if it admits a combinatorial geodesic representative that is Morse. Denote the set of all Morse points in the Roller boundary of Y by $\partial_{R,M} Y$. By [BF18a], there exists a surjective map $\Phi : \partial_{R,M} Y \to \partial_M Y$, which sends any combinatorial geodesic representative $[\gamma] \in \partial_{R,M} Y$ to the point $[\gamma]$ in the Morse boundary represented by the quasi-geodesic γ.

Let $n \in \mathbb{N}_0$. We call two hyperplanes h, h' *n-strongly separated*, if they are disjoint and there are at most n many hyperplanes that intersect both h and h'. For $n = 0$ we simply write *strongly separated*.

In [CS15], Charney and Sultan characterized Morse geodesics in uniformly locally finite CAT(0) cube complexes as follows:

Theorem 2.7 (Theorem 4.2 in [CS15])**.** *Let Y be a uniformly locally finite* CAT(0) *cube complex. There exist $r > 0$, $n \geq 0$ (depending only on D and the maximal valence ν), such that a geodesic ray γ in Y is D-contracting if and only if γ crosses an infinite sequence of hyperplanes $h_1, h_2 \dots$ at points $y_i := \gamma \cap h_i$ satisfying*

(1) h_i*,* h_{i+1} *are* n*-separated and*
(2) $d(y_i, y_{i+1}) < r$*.*

Remark 2.8. Consider a Morse geodesic ray γ in a uniformly locally finite CAT(0) cube complex Y. This geodesic ray induces a consistent orientation of hyperplanes and thus a point $y \in \partial_R Y$. The theorem by Charney-Sultan implies that the component $C \subset \partial_R Y$ that contains y is a bounded cube complex. If it were unbounded, we would find for any $N > 0$ a time T, such that any two hyperplanes crossed by $\gamma|_{[T,\infty)}$, are not N-strongly separated (no hyperplane crossed by γ can intersect C, as γ is a geodesic and moves infinitely far away from any hyperplane it crosses). We conclude that the preimage $\Phi^{-1}([\gamma])$ of Morse points $[\gamma] \in \partial_M Y$ is always a bounded component.

2.3. **Right-angled Artin groups.** Let Γ be a finite, undirected graph with no multiple edges and no loops of length 1. We denote its vertices by V_Γ and its edges by E_Γ. The Right-angled Artin group (RAAG) associated to Γ is defined by

$$A_\Gamma := \langle V_\Gamma | [v_i, v_j] \text{ for all } (v_i, v_j) \in E_\Gamma \rangle$$

We now build a cube complex whose fundamental group is A_Γ. Its universal covering Y_Γ will inherit a cube complex structure and will be CAT(0). Furthermore, A_Γ acts properly, cocompactly by cube-automorphisms on Y_Γ. We start with one vertex and glue both endpoints of V_Γ-many edges to this vertex. We label the edges by the vertices $v_i \in V_\Gamma$. Whenever we have vertices $v_{i_1}, \dots, v_{i_k}$ such that $(v_{i_j}, v_{i'_j}) \in E_\Gamma$, we glue a k-cube along the edges $v_{i_1}, \dots v_{i_k}$ in such a way that the k-cube is glued to become a k-dimensional torus (i. e. parallel edges of the k-cube are all glued to the same edge in the complex). The resulting cube complex is called the Salvetti complex of Γ. We denote its universal covering by Y_Γ. The following statement is proven during the proof of Theorem 5.1 in [CH17].

Proposition 2.9 ([CH17])**.** *Given a finite, undirected graph* Γ*, any contracting geodesic ray in* Y_Γ *crosses an infinite sequence of hyperplanes as in Theorem 2.7, where all hyperplanes in the sequence are strongly separated.*

Remark 2.10. In [Fer18, FLM18], the notion of a regular point was introduced, which can be defined as follows (cf. Proposition 7.5 in [Fer18]): A point $\xi \in \partial_R Y$ is called *regular* if the ultrafilter σ_ξ contains an infinite chain $h_0 \subsetneq h_1 \subsetneq \dots$ such that the corresponding hyperplanes $w_0, w_1 \dots$ are strongly separated. From the results above, it follows that the Morse boundary of a RAAG is contained in the image of the regular points under the map Φ.

3. Comparing $\mathcal{FG}$ and $\mathcal{FQ}$

Let Y be a CAT(0) cube complex and fix a base point $o \in Y$ for the rest of this section. By Cashen-Mackay, the contracting boundary $\partial_M Y$ carries the topologies of fellow-traveling geodesics – $\mathcal{FG}$ – and of fellow-traveling quasi-geodesics – $\mathcal{FQ}$. We will introduce a third topology and then show that it coincides with $\mathcal{FG}$ but differs from $\mathcal{FQ}$ in general.

Let $h_1, \ldots, h_n$ be distinct hyperplanes in Y. Define the set

$$U_{o,h_1,\ldots,h_n} := \{\xi \in \partial_M Y | \text{The unique geodesic representative of } \xi \text{ based at } o \text{ crosses the hyperplanes } h_1, \ldots, h_n.\}.$$

It is easy to see that the collection $\{U_{o,h_1,\ldots,h_n}\}_{n,h_1,\ldots,h_n}$ is a basis of the topology $\mathcal{HYP}$ introduced in section 1.

Proposition 3.1. *Let Y be a uniformly locally finite* CAT(0) *cube complex, X its contracting boundary as a set. Then $\mathcal{FG} = \mathcal{HYP}$ on X.*

Corollary 3.2. *$\mathcal{HYP}$ is independent of the choice of base point.*

Proof of Corollary 3.2. This follows from Proposition 2.2 and Proposition 3.1 □

Proof of Proposition 3.1. Suppose, all cubes in Y have dimension at most n. We start by showing that $\mathcal{FG} \subset \mathcal{HYP}$. Let $\xi \in X$ and $U_{o,R}(\xi)$ an element of the neighbourhood basis of $\mathcal{FG}$. Denote the geodesic representative of ξ starting at o by γ. Let $(h_i)_i$ be the sequence of hyperplanes crossed by γ ordered in the order they are crossed by γ. We find a constant D_ξ such that the geodesic representative of ξ starting at o is D_ξ-contracting. Denote $\kappa := \kappa(D_\xi, 1, 0)$. By [CH17], we find $m \in \mathbb{N}$, $r > 0$ and a subsequence $(h_{i_j})_j$ of $(h_i)_i$ consisting of pairwise disjoint, m-strongly separated hyperplanes such that $d(h_{i_j} \cap \gamma, h_{i_{j+1}} \cap \gamma) < r$. Let h_{i_N} be the first hyperplane in $(h_{i_j})_j$ such that $N > 1$ and $d(o, h_{i_j}) > R + C$, where

$$C := 4\sqrt{n}(m + \sqrt{n}r) + 8\kappa.$$

Consider the basis-element $U_{o,h_{i_{N+2}}}$. Clearly, $\xi \in U_{o,h_{i_{N+2}}}$. We claim that $U_{o,h_{i_{N+2}}} \subset U_{o,R}(\xi)$. Let $\eta \in U_{o,h_{i_{N+2}}}$ and let β be the geodesic representative of η starting at o. Note that, since β is a geodesic, it can cross every hyperplane at most once. Since $\eta \in U_{o,h_{i_{N+2}}}$, β has to cross h_{i_N}, $h_{i_{N+1}}$ and $h_{i_{N+2}}$ at some point.

Denote the restrictions of β and γ to the area between h_{i_j} and $h_{i_{j+1}}$ by β_j and γ_j respectively. We focus our attention on the restriction of β and γ to the area between h_{i_N} and $h_{i_{N+2}}$, i. e. on β_N, β_{N+1}, γ_N and γ_{N+1}. We will estimate the distance between $p := \gamma \cap h_{i_{N+1}}$ and $q := \beta \cap h_{i_{N+1}}$. Since Y has uniformly bounded dimension, we can do so by estimating the number of hyperplanes that separate p from q.

Suppose, $k \in \mathcal{W}(p, q)$. Since β and γ both start at o, k has to be crossed by exactly one of them at some point before the two paths cross $h_{i_{N+1}}$. There are three possibilities.

FIGURE 2. The graph Γ_0 that induces $G_0 := A_{\Gamma_0}$.

Case 1: Suppose, k intersects neither β_N nor γ_N. Then, k intersects h_{i_N}. Since h_{i_N} and $h_{i_{N+1}}$ are m-strongly separated, there can be at most m-many such hyperplanes.

Case 2: Suppose, k intersects γ_N. By assumption, $d(\gamma \cap h_{i_N}, \gamma \cap h_{i_{N+1}}) < r$. Therefore, there can be at most $\sqrt{n}r$-many such hyperplanes.

Case 3: Suppose, k intersects β_N. Focus on the area between $h_{i_{N+1}}$ and $h_{i_{N+2}}$. Since β is a geodesic, k has to intersect either γ_{N+1} or $h_{i_{N+2}}$. Again, there are at most $m + \sqrt{n}r$-many such hyperplanes.

We conclude that, in total, there are at most $2(m + \sqrt{n}r)$-many hyperplanes that separate p from q. Thus, $d(p, q) \leq 2\sqrt{n}(m + \sqrt{n}r)$. Lemma 4.6 in [CM18] now implies that, there exists a point $q' \in \beta$ which is κ-close to $\gamma \setminus B_R(o)$. This relies on our choice of C which was made specifically to suit Cashen-Mackays result. This implies that $\eta \in U_{o,R}(\xi)$. Therefore, $\xi \in U_{o,k_{i_N}} \subset U_{o,R}(\xi)$ which implies that $U_{o,R}(\xi) \in \mathcal{HYP}$.

We are left to show that $\mathcal{HYP} \subset \mathcal{FG}$. Let $h_1, \ldots, h_n$ be distinct hyperplanes and $\xi \in U_{o,h_1,\ldots,h_n}$. Let γ be the geodesic representative of ξ based at o. We find a constant D_ξ such that γ is D_ξ-contracting. Choose $R > 0$ sufficiently large, such that $d(h_i, \gamma \setminus B_R(o)) > \kappa(D_\xi, 1, 0)$ for all i. Such R exists, since a geodesic γ cannot stay uniformly close to any hyperplane it crosses (for example because the distance function $d(\gamma(t), \beta(t))$ of two geodesics in a CAT(0) space is convex). Clearly, $\xi \in U_{o,R}(\xi)$ and we need to show that $U_{o,R}(\xi) \subset U_{o,h_1,\ldots,h_n}$.

Let $\eta \in U_{o,R}(\xi)$ and β the geodesic representative of η based at o. Then there exists a point $p \in \beta$, such that $d(p, \gamma \setminus B_R(o)) \leq \kappa(D_\xi, 1, 0)$. In particular, if $\gamma_s \in \gamma \setminus B_R(o)$ satisfies $d(p, \gamma) = d(p, \gamma_s)$, then the geodesic δ from p to γ_s is completely contained in the $\kappa(D_\xi, 1, 0)$-neighbourhood of $\gamma \setminus B_R(o)$. Since $d(h_i, \gamma \setminus B_R(o)) > \kappa(D_\xi, 1, 0)$, δ cannot intersect h_i for any i. We conclude that, for all i, h_i separates o from p, which implies that β crosses h_i for all i. Therefore, $\eta \in U_{o,h_1,\ldots,h_n}$ and $\xi \in U_{o,R}(\xi) \subset U_{o,h_1,\ldots,h_n}$, which completes the proof. □

In contrast to Proposition 3.1, $\mathcal{HYP}$ and $\mathcal{FQ}$ do not coincide in general. Consider the right-angled Artin group $G_0 := A_{\Gamma_0}$ induced by the graph Γ_0 depicted in figure 2. Its Salvetti complex can be obtained as follows. Consider three distinct tori and consider two simple closed curves in each as depicted in figure 3. Glueing the curves b in the first two tori together and glueing the curves c in the second two tori together, we obtain the Salvetti complex of G_0. Denote its universal covering by $Y_0 := Y_{\Gamma_0}$.

Proposition 3.3. *If $X = \partial_M Y_0$ for Γ_0 as in figure 2, then $\mathcal{HYP} \subsetneq \mathcal{FQ}$ on X.*

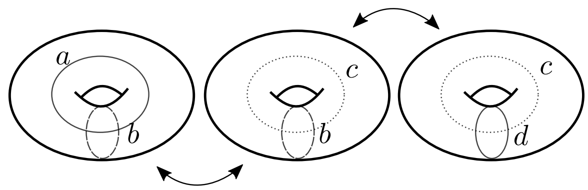

FIGURE 3. The Croke–Kleiner example

Proof of Proposition 3.3. By [CM18], we know that $\mathcal{FG} \subset \mathcal{FQ}$, so we are left to show that $\mathcal{HYP} \neq \mathcal{FQ}$. In particular, we need to show that there exists some neighbourhood $U_{o,R}(\xi)$ such that there exists no $h_1, \ldots, h_n$ such that $\xi \in U_{o,h_1,\ldots,h_n} \subset U_{o,R}(\xi)$.

We recall a few facts about the space Y_0. In [CK00], Croke and Kleiner showed that Y_0 can be written as a union of the following 'blocks': Consider two of the tori in the Salvetti complex that are identified along a curve (i. e. the first and second torus, or the second and third torus). The preimage of this subspace under the projection from the universal covering is a disjoint union with each component being isometric to the product of a 4-valent tree with $\mathbb{R}$. Furthermore, the preimage of any of the three tori is a disjoint union of 2-dimensional flats. We denote the collection of flats corresponding to the first torus by $\mathcal{A}$, the collection corresponding to the second torus by $\mathcal{B}$ and the collection corresponding to the third torus by $\mathcal{C}$. The Salvetti complex of Γ has four hyperplanes. Denote the collection of preimages of the hyperplane crossed by the curve a by A, the collection of preimages of the hyperplane crossed by the curve b by B, the collection of preimages of the hyperplane crossed by the curve c by C and the collection of preimages of the hyperplane crossed by the curve d by D. Thus, the collection of hyperplanes in Y_0 is the disjoint union of A, B, C, D.

Before we start with the main construction, we remark that, whenever we concatenate paths in the following proof, we do not rescale their parametrisation. We adjust their parametrisation by time-shift to allow concatenation, but they remain parametrised by arc-length.

We will introduce a specific point $\xi \in \partial_M Y_0$ and find some neighbourhood $U_{o,R}(\xi)$ as described above (see also figure (4)). Fix a base point $o \in Y_0^{(0)}$ and choose edges, incident to o, that induce hyperplanes $b_0 \in B$ and $c_1 \in C$ respectively. Consider the unique geodesic segment that starts at o, has length $\sqrt{2}$ and crosses both b_0 and c_1 at an angle of $\frac{\pi}{4}$ (i.e. it crosses the square spanned by b_0 and c_1 along the diagonal). Denote this geodesic segment by γ_1 and its second endpoint by w_1. Note that both endpoints of γ_1 are vertices in Y_0. There exists a unique flat $F \in \mathcal{B}$ that contains w_1.

The geodesic segment γ_1 has to possible geodesic extension of length $\sqrt{2}$ in F. Choose one of these extensions and denote it by γ_2. The endpoints of γ_2 are w_1 and a point we denote w_2.

We continue the construction of γ as follows. Let $\mathcal{F}_i \subset \{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$ be the periodic sequence defined by the period

$$\mathcal{B}, \mathcal{C}, \mathcal{B}, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{B}, \mathcal{A}.$$

By the construction above, the geodesic segment γ_1 is contained in a flat $F_1 \in \mathcal{F}_1$ and the geodesic segment γ_2 is contained in a flat $F_2 \in \mathcal{F}_2$. Suppose we had constructed geodesic segments γ_j for $1 \leq j \leq i$ which can be concatenated to a geodesic segment in Y_0. We denote the endpoints of γ_j by w_{j-1} and w_j. There exists a unique flat $F \in \mathcal{F}_{i+1}$ that contains the point w_i. The geodesic segment γ_i has two possible geodesic extensions of length $\sqrt{2}$ in F. Choose one of these extensions and denote it by γ_{i+1}. The endpoints of γ_{i+1} are w_i and a point we denote w_{i+1}.

This construction provides us with a geodesic ray γ in Y_0, which is the concatenation of all the γ_i. Due to the form of the sequence $\mathcal{F}_i$, γ stays in each block for at most distance $3\sqrt{2}$. Therefore, γ is sublinearly contracting and defines a point $\xi \in \partial_M Y_0$. Note that γ crosses at most three hyperplanes in every flat it crosses.

The image of γ crosses an infinite collection of flats $(F_l)_l$, with $F_l \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ ordered in the way they are crossed by γ. Indeed, $\gamma_l = \gamma \cap F_l$, where we identify any path segment with its image.

Let $R \gg 0$ and pick any finite collection of hyperplanes $h_1, \ldots, h_n$ each of which is separating o from ξ. Since γ is contracting, we find a sequence of strongly separated hyperplanes $(k_j)_j$ as before. In particular, we find a hyperplane $k := k_N$ such that for all i, k separates ξ from h_i. We have $\xi \in U_{o,k} \subset U_{o,h_1,\ldots,h_n}$. We will construct a $(8\sqrt{2}, 1)$-quasi-geodesic β'_k such that $[\beta'_k] \in U_{o,k} \setminus U_{o,R}(\xi)$.

The main step of the proof is the construction of an auxiliary quasi-geodesic β, which will be a path in the 1-skeleton of Y_0, parametrized by unit speed on every edge. In particular, it will be determined completely by the ordered sequence of hyperplanes it crosses. We will write β as the concatenation of paths $(\beta_l)_{l\in\mathbb{N}}$, where $\beta_l \subset F_l$. In what follows, whenever we consider a path $p : [a, b] \to Y_0$, we will refer to $p(b)$ as its endpoint.

Choose an integer $3 < \Delta < R$. Let $b'_0 \in B$ be the unique hyperplane in B that is not equal to b_0 and has minimal distance to o. Consider the geodesic starting at o and crossing through b'_0 and extend this geodesic by $\Delta + 3$ many steps in the flat F_1. Denote this geodesic line by p_1. Define q_1 to be the geodesic from the endpoint of p_1 to its closest point projection onto $F_1 \cap F_2$. Note that this geodesic is completely contained in the 1-skeleton of Y_0, as the endpoint of p_1 is separated from $F_1 \cap F_2$ only by hyperplanes in C. Denote $\beta_1 := p_1 * q_1$. Note that β_1 is a $(\sqrt{2}, 0)$-quasi-geodesic because the l^1-metric on the euclidean plane is $\sqrt{2}$-bi-Lipschitz-equivalent to the euclidean metric

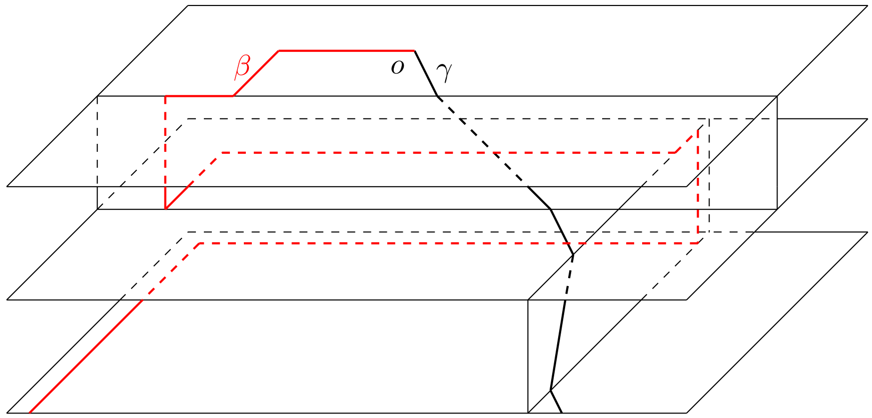

FIGURE 4. The concatenation β of the quasi-geodesics β_k (red) and the geodesic γ (black). The scale of β is reduced for presentation.

on the plane. Further, the endpoint of β_1 is separated from γ by $\Delta+3$-many hyperplanes that pairwise don't intersect. Therefore, the distance of the endpoint of β_1 to any point in γ is at least $\Delta + 3$.

We now provide the general definition of β_l by an inductive procedure. Suppose, we have defined $(\sqrt{2}, 0)$-quasi-geodesics $\beta_k \subset F_k$ for all $k < l$. Additionally, suppose that for all $1 < k < l$, the distance of any point in β_k to any point in γ is at least Δ. Further, assume that for all $1 \leq k < l$, the endpoint of β_k lies in $F_k \cap F_{k+1}$ and has distance at least $\Delta + 3$ from any point in γ.

For all $k \geq 1$, denote the endpoint of β_k by v_k and recall that the endpoint of γ_k is denoted by w_k. Define

$$M_l := d(v_{l-1}, F_l \cap F_{l+1}) = \#\{\text{hyperplanes separating } v_{l-1} \text{ from } F_l \cap F_{l+1}\}$$

and

$$N_l := \max\left(\Delta + 3, 5M_l, 2\sum_{k<l} l(\beta_k)\right).$$

Note that this implies

$$\frac{N_l}{2} - M_l \geq \frac{N_l}{4} + \frac{M_l}{8}. \tag{1}$$

We distinguish between three cases:

Case 1: If $F_l \in \mathcal{A}$, then $F_{l-1} \in \mathcal{B}$. In this case, the 1-dimensional cubically embedded line $F_{l-1} \cap F_l$ is only crossed by hyperplanes in B. Since v_{l-1} and w_{l-1} both lie in $F_{l-1} \cap F_l$, we conclude that they are only separated by hyperplanes in B. In particular, since the distance between them has to be at least $\Delta + 3$ by the assumption of the induction, there are at least $\Delta + 3$ many hyperplanes separating v_{l-1} and w_{l-1}.

By construction, γ_l crosses one hyperplane in B and one hyperplane in A. Define p_l to first cross the unique hyperplane in B adjacent to v_{l-1} that does not separate v_{l-1} from w_{l-1}. Extend p_l by $N_l - 1$ many steps in F_l. Define q_l to be the path starting at the endpoint of p_l and crossing the hyperplane in A that is crossed by γ_l. Define $\beta_l := p_l * q_l$.

We now prove that β_l satisfies the assumptions of the induction. Clearly, it is a $(\sqrt{2}, 0)$-quasi-geodesic in F_l. Further, since γ crosses at most three hyperplane in every flat and v_{l-1} and w_{l-1} are separated by at least $\Delta + 3$ many hyperplanes, any point in β_l is separated by at least Δ many hyperplanes in B from any point in γ. Since no two hyperplanes in B intersect, this implies that the distance between any point in β_l and any point in γ is at least Δ. Further, the endpoint of β_l is separated by $\Delta+3$ many additional pairwise non-intersecting hyperplanes from any point in γ. We conclude that the distance of the endpoint of β_l to any point in γ is at least $\Delta + 3$.

Finally, we show that the endpoint of β_l lies in $F_l \cap F_{l+1}$. Since β_l starts in F_l and crosses only hyperplanes that intersect with F_l, it lies in F_l. Given a vertex $v \in F_l$, it lies in $F_l \cap F_{l+1}$ if and only if there exists a point $w \in F_{l+1}$ such that no hyperplane in A separates v from w. This is the case for the endpoints v_l, w_l of β_l and γ_l and we know that w_l lies in F_{l+1} by definition of γ and F_l. We conclude that β_l satisfies all the assumption of the induction.

Case 2: If $F_l \in \mathcal{C}$, then we do the same thing as in case 1, except that the roles of the sets A and B are played by the sets D and C respectively.

Case 3: If $F_l \in \mathcal{B}$, assume without loss of generality that $F_{l-1} \in \mathcal{A}$. Again, the case of $F_{l-1} \in \mathcal{C}$ can be obtained by role-swapping of A with D and of B with C. We know that v_{l-1}, $w_{l-1} \in F_{l-1} \cap F_l$ and they are separated by at least $\Delta + 3$ many hyperplanes in B. By construction of γ, there is a unique hyperplane in C that is adjacent to w_{l-1} and is not crossed by γ_l. Define p_l to start at w_{l-1}, to cross this unique hyperplane and extend it by $N_l - 1$ many steps in F_l. Define q_l to start at the endpoint of p_l and to cross all hyperplanes in B that separate that endpoint from the line $F_l \cap F_{l+1}$. Note that these are all hyperplanes that separate the endpoint of p_l from $F_l \cap F_{l+1}$, as all hyperplanes crossed by p_l are also crossed by $F_l \cap F_{l+1}$. Define $\beta_l := p_l * q_l$.

Again, we show that β_l satisfies all the assumption of the induction. It is a $(\sqrt{2}, 0)$-quasi-geodesic by construction. By the same argument as before, there are Δ-many hyperplanes in B that separate any point of p_l from any point on γ. Furthermore, there are $N_l \geq \Delta + 3$ many hyperplanes that separate any point on q_l from any point on γ, specifically, the hyperplanes crossed by p_l. We conclude that β_l satisfies the assumption of the induction.

Note that for any $l > 1$, there is no hyperplane crossed by both p_{l-1} and p_l. Together with the fact that q_{l-1} never crosses hyperplanes of the same type as p_{l-1} and p_l, this implies in particular that $p_{l-1} * q_{l-1} * p_l$ is a combinatorial geodesic.

Claim 1. The path β is an $(8\sqrt{2}, 1)$-quasi-geodesic.

Before we prove the claim, we point out that β is not contracting. We will construct suitable contracting paths β'_k from it, once we have proven the claim.

Proof of Claim. Let $0 \leq t < s$. Since β is a concatenation of geodesic lines, we have $d(\beta(t), \beta(s)) \leq |s-t|$. We are left to show that $d(\beta(t), \beta(s)) \geq \frac{1}{8\sqrt{2}}|s-t| - 1$. Suppose, $\beta(s) \in F_l$. Then $\beta(t) \in F_k$ for some $k \leq l$. There are three cases.

Case 1: If $k = l$, we find t', s' such that $t - s = t' - s'$ and $\beta(t) = \beta_l(t')$, $\beta(s) = \beta_l(s')$. We have already seen that β_l is a $(\sqrt{2}, 0)$-quasi-geodesic by construction.

Case 2: Suppose $k = l - 1$. We first note that β is a concatenation of paths β_l, which in turn are concatenations of geodesic lines p_l, q_l that lie in the 1-skeleton of Y_0. Each of these geodesic lines crosses hyperplanes from only one of the four families A, B, C, D. If we write down for every geodesic line, which type of hyperplanes it crosses, in order of concatenation, we get a periodic sequence with period

$$C, B, C, D, B, C, B, A.$$

In other words, p_1 crosses hyperplanes in C, q_1 crosses hyperplanes in B, p_2 crosses hyperplanes in C, q_2 crosses hyperplanes in D, etc. following the periodic sequence described above.

We will distinguish between several subcases.

Case 2a: Suppose, $\beta(s) \in p_l$ and $\beta(t) \in q_{l-1}$. By construction, the concatenation $q_{l-1} * p_l$ is either a geodesic, or contained inside the 1-skeleton of the flat F_{l-1}, hence a $(\sqrt{2}, 0)$-quasi-geodesic.

Case 2b: Suppose, $\beta(s) \in p_l$ and $\beta(t) \in p_{l-1}$. As we noted before stating the claim, the concatenation $p_{l-1} * q_{l-1} * p_l$ is a combinatorial geodesic for all $l > 1$. Thus, we have

$$\begin{aligned} d(\beta(s), \beta(t)) &\geq \frac{1}{\sqrt{2}} d_{l^1}(\beta(s), \beta(t)) \\ &= \frac{1}{\sqrt{2}}|s-t|. \end{aligned}$$

Case 2c: Suppose, $\beta(s) \in q_l$ and $\beta(t) \in q_{l-1}$. By checking the type of hyperplanes crossed by q_{l-1}, p_l and q_l, we see that $q_{l-1} * p_l * q_l$ is a combinatorial geodesic and thus a $(\sqrt{2}, 0)$-quasi-geodesic by the same argument as in Case 2b.

Case 2d: Suppose, $\beta(s) \in q_l$ and $\beta(t) \in p_{l-1}$. By definition of N_l, we know that p_l crosses at least twice as many hyperplanes as all β_k for $k < l$

together. We conclude that $\beta(s)$ is separated from $\beta(t)$ by at least $\frac{N_l}{2} - M_l$ many hyperplanes and $|s-t| \leq 2N_l + M_l$. Therefore, using equation (1), we have

$$\begin{aligned} d(\beta(s), \beta(t)) &\geq \frac{1}{\sqrt{2}} d_{l^1}(\beta(s), \beta(t)) \\ &\geq \frac{1}{\sqrt{2}} \left(\frac{N_l}{2} - M_l \right) \\ &\geq \frac{1}{8\sqrt{2}} (2N_l + M_l) \\ &\geq \frac{1}{8\sqrt{2}} |s-t|. \end{aligned}$$

Combining cases 2a-d, we conclude that $p_{l-1} * q_{l-1} * p_l * q_l$ is a $(8\sqrt{2}, 0)$-quasi-geodesic.

Case 3: Suppose $k < l-1$. If $\beta(s) \in q_l$, the argument from Case 2d applies and thus, the $(8\sqrt{2}, 0)$-quasi-geodesic inequalities are satisfied.

If $\beta(s) \in p_l$, let $\beta(T)$ be the end point of p_{l-1}. Clearly, $\beta(s)$ and $\beta(T)$ are separated by at least $s - T - \frac{1}{2}$ many hyperplanes. Furthermore, p_{l-1} provides at least $\frac{N_{l-1}}{2}$ many additional hyperplanes that separate $\beta(s)$ from any $\beta(t) \in \beta_k$ for $k < l-1$, as p_{l-1}, q_{l-1} and p_l cross mutually disjoint families of hyperplanes. Using the fact that $|T-t| \leq 2N_{l-1}$ by definition of N_{l-1}, we conclude that

$$\begin{aligned} d(\beta(s), \beta(t)) &\geq \frac{1}{\sqrt{2}} d_{l^1}(\beta(s), \beta(t)) \\ &\geq \frac{1}{\sqrt{2}} \left(\frac{N_{l-1}}{2} + s - T - \frac{1}{2} \right) \\ &\geq \frac{1}{8\sqrt{2}} \left| T - t + s - T - \frac{1}{2} \right| \\ &= \frac{1}{8\sqrt{2}} |s-t| - \frac{1}{16\sqrt{2}}. \end{aligned}$$

We conclude that β satisfies the $(8\sqrt{2}, 1)$-quasi-geodesic inequalities for any $0 \leq t \leq s$, which concludes the proof of the claim □

We now return to our open neighbourhood $U_{o,k}$ of ξ. Since every hyperplane in Y_0 is contained in one block of Y_0, it can either intersect flats in $\mathcal{A} \cup \mathcal{B}$ or in $\mathcal{B} \cup \mathcal{C}$. Without loss of generality, k only intersects flats in $\mathcal{A} \cup \mathcal{B}$. Since γ crosses infinitely many flats of all three types, there has to be an L, such that k separates o from F_L. In particular, the concatenation $\beta_1 * \cdots * \beta_L$ crosses k. We can extend this concatenation to a contracting $(8\sqrt{2}, 1)$-quasi-geodesic β'_k, which provides us with a point $\eta_k := [\beta'_k] \in \partial_M Y_0$. Note that $[\beta'_k] \neq [\beta]$, in fact, β is not a contracting quasi-geodesic. However, β'_k is a quasi-geodesic with the property that $\beta'_k \cap N_\Delta(\gamma \setminus B_R(o)) = \emptyset$ since $R > \Delta$. If we choose $\Delta > \kappa(\rho_\xi, 8\sqrt{2}, 1)$, we conclude that $\eta_k \in U_{o,k} \setminus U_{o,R}(\xi)$. In summary: Given $R \gg \kappa(\rho_\xi, 8\sqrt{2}, 1)$ and any hyperplane k, we have found a point $\eta_k = [\beta'_k] \in U_{o,k} \setminus U_{o,R}(\xi)$. This implies that $\mathcal{HYP} \neq \mathcal{FQ}$, which completes the proof of the proposition. □

Remark 3.4. Note that we can chose Δ rather freely in our construction. In particular, it is not possible to adapt the number $\kappa(\rho, K, C)$ in Cashen-Mackays definition of fellow-traveling to make the constructed paths β'_k fellow-travel along γ.

Remark 3.5. The construction in the proof of Proposition 3.3 can be done for most points in the Morse boundary of Y_0, although the construction becomes more messy, as the geodesic γ becomes more complicated. We see that $\mathcal{FQ}$ and $\mathcal{FG}$ provide different open neighbourhoods around nearly every point of $\partial_M Y_0$.

4. Defining a metric on the Morse boundary of Right-angled Artin groups

Let Γ be a finite graph with no multiple edges and no self-loops for the remainder of this section. Denote by A_Γ the induced RAAG and by Y_Γ the universal covering of the Salvetti complex. More generally, let Y be a locally finite CAT(0) cube complex satisfying the following property:

($\diamondsuit$) For every Morse geodesic ray γ, there exists $r \geq 0$ and an infinite family of strongly separated hyperplanes h_i crossed by γ, such that for $p_i := \gamma \cap h_i$ we have $d(p_i, p_{i+1}) \leq r$.

We will show that, whenever Y satisfies ($\diamondsuit$), the topology $\mathcal{HYP}$ on its Morse boundary is metrizable and has a nice description. The metric will depend on the choice of a base point o, so we will obtain a family of metrics d_o. In fact, these metrics even induce different cross ratios on the Morse boundary, as we will see further below.

Before we define the actual metric, recall the following fact, which is an essential part of the proof of the Urysohn metrization theorem (cf. [Roy88]).

Let $(Y, \mathcal{T})$ be a topological space, $\{U_n\}$ a countable basis of $\mathcal{T}$ and $f_n : Y \to [0, e^{-n}]$ continuous maps such that $\mathrm{supp}(f_n) \subset U_n$ and for every $y \in Y$, $U \in \mathcal{T}$ with $y \in U$ there exists some f_n with $f_n(y) \neq 0$ and $\mathrm{supp}(f_n) \subset U$. Then the map

$$d(x, y) := \sup_n(|f_n(x) - f_n(y)|)$$

is a metric and its induced topology is equal to $\mathcal{T}$. In fact, we can even have finitely many sets $U_{n,1}, \dots, U_{n,l_n}$ and finitely many functions $f_{n,1}, \dots f_{n,l_n} : Y \to [0, e^{-n}]$ with the properties above and the construction still yields a metric that induces $\mathcal{T}$. We will explicitly choose the sets U_n and construct functions f_n. We start by proving certain properties that will justify our choice.

Lemma 4.1. *If Y is a locally finite* CAT(0) *cube complex satisfying* ($\diamondsuit$), *then the family $\{U_{o,k}\}_{k\in\mathcal{W}(Y)}$ is a countable basis of $\mathcal{HYP}$. Furthermore, $U_{o,k}$ is open and closed in $\mathcal{HYP}$.*

Proof. The countability of $\{U_{o,k}\}_k$ follows from the local finiteness of Y. Next, we will show that $\{U_{o,k}\}_k$ is a basis. Let $h_1, \dots, h_n$ be a collection of hyperplanes such that $U_{o,h_1,\dots,h_n} \neq \emptyset$. Let $\xi \in U_{o,h_1,\dots,h_n}$ and $\gamma \in \xi$ a geodesic representative based at o. By Proposition 2.9, we find a sequence of strongly separated hyperplanes $(k_i)_i$ that are crossed by γ. Consider the first hyperplane k_i that is crossed by γ after it has crossed all h_j. Since all k_i are strongly separated, k_{i+1} cannot cross h_j for any j and thus, $\xi \in U_{o,k_{i+1}} \subset U_{o,h_1,\dots,h_n}$. We conclude that $\{U_{o,k}\}_k$ is a basis.

Finally, we show that $U_{o,k}$ is closed in $\mathcal{HYP}$. Let $\xi \in \partial_M Y \setminus U_{o,k}$ and $\gamma \in \xi$ the geodesic representative based at o. There exists a sequence of strongly separated hyperplanes $(k_i)_i$ that is crossed by γ. Since γ does not cross k, there can be at most one k_i that crosses k. Let k_I be the first hyperplane in $(k_i)_i$ after the one that crosses k (pick any k_I if no k_i crosses k). We have that $\xi \in U_{o,k_I} \subset \partial_M Y \setminus U_{o,k}$. We conclude that $U_{o,k}$ is closed. □

Lemma 4.1 allows us to make the construction from Urysohns metrization theorem very explicit. We can choose $\{U_{o,k}\}_k$ as our countable basis and define

$$f_k(\xi) := \begin{cases} e^{-\#\mathcal{W}(o,k)} & \text{if } \xi \in U_{o,k}, \\ 0 & \text{else} \end{cases}$$

Recall that $\mathcal{W}(o,k)$ is the set of hyperplanes that separates o from k. Then, by the proof of Urysohns metrization theorem,

$$d_o(\xi,\eta) := \sup_k(|f_k(\xi) - f_k(\eta)|)$$

is a metric and its induced topology is equal to the topology generated by $\{U_{o,k}\}_k$, i. e. $\mathcal{HYP}$.

There is a more convenient way to describe d_o. Let $\xi, \eta \in \partial_M Y$, $o \in Y^{(0)}$. Since $f_k(\xi) \in \{0, e^{-\#\mathcal{W}(o,k)}\}$, we see that $f_k(\xi) - f_k(\eta) = 0$ unless one of the two lies in $U_{o,k}$ and the other one does not. Therefore, if we define

$$\mathcal{W}(\xi,\eta) := \{h \in \mathcal{W}(Y) | \text{ Exactly one geodesic representative of } \xi \text{ and } \eta \text{ based at } o \text{ crosses } h\},$$

we have

$$d_o(\xi,\eta) = e^{-\inf\{\#\mathcal{W}(o,k) | k\in\mathcal{W}(\xi,\eta)\}}.$$

To abbreviate, we define

$$[\xi|\eta]_o := \inf(\#\mathcal{W}(o,k) | k \in \mathcal{W}(\xi,\eta))$$

and obtain

$$d_o(\xi,\eta) = e^{-[\xi|\eta]_o}.$$

We use the expression $[\xi|\eta]_o$ as an analogue of the Gromov product in hyperbolic spaces. Note that $[\xi|\eta]_o$ is very different from the product $(\xi|\eta)_o := \#\mathcal{W}(o, \{\xi, \eta\})$ which was used in [BFIM18].

We define the cross ratio of four points w, x, y, z with respect to $[\cdot|\cdot]_o$ by

$$cr_o(w, x, y, z) := [w|x]_o + [y|z]_o - [w|y]_o - [x|z]_o.$$

In addition, we define

$$[w, x, y, z] := (w|x)_o + (y|z)_o - (w|y)_o - (x|z)_o,$$

which is the base point-independent cross ratio introduced in [BFIM18]. Cross ratios often appear on boundaries in a very natural way (cf. [Pau96] and [BFIM18]). One particular desirable feature of cross ratios on boundaries is that they should be independent of the choice of any base point (in contrast to the construction of metrics). The example below shows us that:

1) The difference $|cr_o(\cdot,\cdot,\cdot,\cdot) - [\cdot,\cdot,\cdot,\cdot]|$ is unbounded.
2) The cross ratio cr_o is not independent of the choice of o.

Example 4.2. Consider the 'tree of 3-dimensional flats' that corresponds to the RAAG

$$\mathbb{Z}^3 * \mathbb{Z} = \langle a, b, c, d | [a, b] = [b, c] = [a, c] = 1 \rangle.$$

Denote the Salvetti complex that belongs to the graph $\Gamma = K_3 \cup \{d\}$ by Y. The Cayley-graph of the representation given above can be embedded into Y by an embedding that respects the cube complex structure. Let o be the image of $1 \in \mathbb{Z}^3 * \mathbb{Z}$ under a chosen embedding. The choice of o allows us to represent elements in the visual boundary by infinite words in a, b, c, d and their inverses. Put

$$\begin{aligned} w_n &:= a^n d^\infty, \\ x_n &:= a^n b d^\infty, \\ y &:= a^{-1} b^{-1} d^\infty, \\ z &:= a^{-1} b^{-1} c d^\infty. \end{aligned}$$

Clearly,

$$[w_n|x_n]_1 = [w_n|y]_1 = [w_n|z]_1 = [x_n|z]_1 = [y|z]_1 = 0,$$

while

$$\begin{gathered} (w_n|x_n)_1 = n \\ (w_n|y)_1 = (w_n|z)_1 = (x_n|z)_1 = (y|z)_1 = 0. \end{gathered}$$

We see that

$$\begin{gathered} cr_1(w_n, x_n, y, z) = 0 \\ [w_n, x_n, y, z] = n. \end{gathered}$$

This proves 1). We see that the metric d_o provides us with a cross ratio that is very different from the cross ratio introduced in [BFIM18]. Furthermore, changing the base point also changes the cross ratio cr_o. For example,

$$cr_{c^{-m}}(w_n, x_n, y, z) = 0 + m - 0 - 0 = m \neq cr_1(w_n, x_n, y, z).$$

This proves 2).

5. $\mathcal{HYP}$ and the visual topology

In this section, we will connect the topologies $\mathcal{HYP}$ and $\mathcal{FG}$ with the visual topology on the Morse boundary of CAT(0) cube complexes and the quotient topology coming from the Roller boundary. All together, this will provide us with a new way to tackle the question whether the subspace topology of the visual topology on the Morse boundary is a quasi-isometry-invariant. In particular, our understanding of various cubulable groups provides us with many examples to study this question with. We start by noting a result by Cashen-Mackay and one by Beyrer-Fioravanti.

Lemma 5.1 (Proposition 7.3 from [CM18])**.** *Let Y be a* CAT(0) *space. Then $\mathcal{FG}$ agrees with the subspace topology induced by the visual topology on $\partial_M Y$.*

If Y is a locally finite CAT(0) cube complex, the Roller boundary also induces a topology on $\partial_M Y$. Specifically, consider the projection map $\Phi : \partial_{R,M} Y \to \partial_M Y$ introduced in section 2.2. The Roller boundary induces a subspace topology on $\partial_{R,M} Y$. The following result is part of Theorem 3.10 in [BF18a].

Theorem 5.2 ([BF18a])**.** *Let Y be a uniformly locally finite* CAT(0) *cube complex and equip $\partial_M Y$ with the visual topology. Then the map $\Phi : \partial_{R,M} Y \to \partial_M Y$ is surjective and continuous.*

Furthermore, the map $\phi : \partial_{R,M} Y \diagup \sim \to \partial_M Y$ where asymptotic points in $\partial_{R,M} Y$ are identified and the quotient is equipped with the quotient topology, is a homeomorphism.

The topology on the Roller boundary used in this Theorem is connected to several rigidity results (cf. [BFIM18], [BF18b]). However, these rigidity results use the cross ratio $[\cdot, \cdot, \cdot, \cdot]$, which is different from cr_o, as we have seen in section 4.

Combining Theorem 1.1, Lemma 5.1 and Theorem 5.2, we obtain Corollary 1.3. This gives us a new approach to tackle the question, whether the Morse boundary with the visual topology is a quasi-isometry invariant for uniformly locally finite CAT(0) cube complexes that admit a cocompact action by isometries. For spaces, where $\mathcal{FG} = \mathcal{FQ}$, this follows from the quasi-isometry invariance of $\mathcal{FQ}$. Theorem 1.2 provides an example, where this is not the case (and suggests the existence of many others). We finish by presenting the naive attempt to prove quasi-isometry invariance of the topology $\mathcal{HYP}$ and by illustrating an obstruction to this invariance.

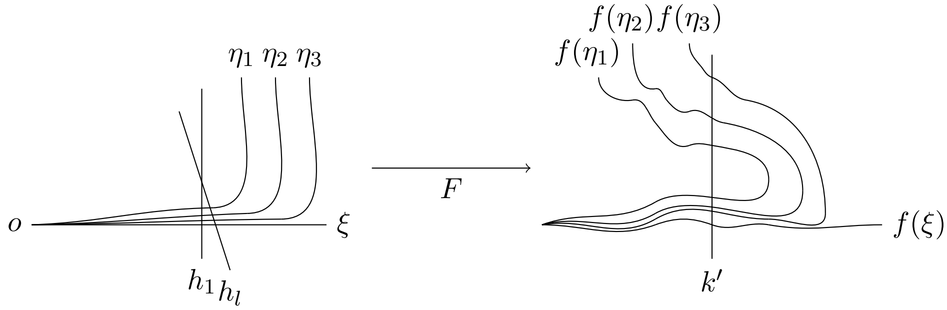

FIGURE 5. Is there a quasi-isometry F such that the images of fellow-traveling, contracting geodesics look like this?

Let Y, Y' be two CAT(0) cube complexes with dimension uniformly bounded by n, X and X' their respective Morse boundaries, $F : Y \to Y'$ a (K, C)-quasi-isometry between them and $f : X \to X'$ the induced bijection of the Morse boundaries. Pick a base point $o \in Y$ and let o' be a vertex closest to $F(o)$. Let k' be a hyperplane in Y' inducing a open set $U_{o',k'} \subset X'$ and suppose, $f(\xi) = \zeta \in U_{o',k'}$. To prove continuity of f, we need to find hyperplanes $h_1, \ldots, h_l$ such that $\xi \in U_{o,h_1,\ldots,h_l}$ and $f(U_{o,h_1,\ldots,h_l}) \subset U_{o',k'}$.

Consider the unique geodesic representative γ of ξ based at o. Concatenating the geodesic segment from o' to $F(o)$ with $F \circ \gamma$ provides us with a $(K, C + \frac{1}{2}\sqrt{n})$-quasi-geodesic representative of $f(\xi)$ which – by assumption – crosses the hyperplane k'. We need to show that for $\eta \in U_{o,h_1,\ldots,h_l}$, the geodesic representative of $f(\eta)$ based at o' crosses k'. For this, consider the geodesic representative δ of η in Y based at o. Since γ is Morse, we can choose $h_1, \ldots, h_l$ such that for all $\eta \in U_{o,h_1,\ldots,h_l}$ its geodesic representative δ stays close to γ for a long distance. While this implies that the image $F \circ \delta$ crosses the hyperplane k' (if δ and γ fellow-travel for sufficiently long), it is not obvious at all that $F \circ \delta$ does not travel back and crosses k' again, implying that the geodesic representative of $f(\eta)$ does not cross k' at all (cf. Figure 5). Note that, if $F \circ \delta$ does travel back, it cannot stay close to $F \circ \gamma$ while doing so, as it is a quasi-geodesic whose constants are controlled by F.

Proving that f is continuous proves that $h_1, \ldots, h_l$ can be chosen such that this kind of back-traveling does not occur. A quasi-isometry that exhibits such back-traveling would provide a counter-example to quasi-isometry invariance. Thus, we finish with the following

Question. Is there a quasi-isometry between 'nice' CAT(0) cube complexes displaying the 'back-traveling' described above?

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